If one examines the solution of Eq. (5), it is found to be one in which the dissipation and production are in balance in the *dissipation* equation. The details are omitted here, but if $C_2 \neq C_1$ this balance does not carry over to the K equation, so that the last two terms are out of balance in Eq. (1). Thus,

$$K \propto a/Y$$

near the body, where a is some nonzero variable. A logarithmic singularity in K is therefore produced which could be matched by no physically correct inner solution near the stagnation point. The conclusion is inescapable: *One or both of the "constants" C_2 and C_1 are incorrect for stagnation flows.* Of the two, C_2 has the most direct support, being selected on the basis of decaying isotropic turbulence. The constant C_1 is usually quoted on the basis of "numerical optimization" by comparison of several calculated flows with experiment.[6] It is the judgment herein that C_1 should be selected equal to C_2 for stagnating flows and that C_2 should retain its conventional value.

It will not be pursued whether or not C_1 could be "fixed" to be a variable dependent upon the local strain rate, for example. Such an endeavor would be warranted in the future. For now, the only statement that will be made is that the conventional vlaue must be changed for stagnating flows. With this change, Eqs. (1) and (2) with Eq. (6) say that as the stagnation point is approached, production and dissipation for both K and E come into balance.

For the case $C_2 = C_1$, a first integral exists for Eqs. (1) and (2):

$$K = z^{1/(1-C_2)} \tag{7}$$

This equation gives an immediate link between the stagnation point condition and the "freestream" condition for

$$K_s = k_s/k_0 = z_s^{1/(1-C_2)} = \left(\frac{1}{2C_\mu^{1/2} L\gamma}\right)^{1/(1-C_2)} \tag{8}$$

Notice in Eq. (8) that since L contains k_0, k_s and k_0 are not simply proportional to each other. In fact,

$$k_s \propto k_0^{C_2/(C_2-1)} \approx k_0^2$$

according to this solution.

Conclusion

For low-speed, two-dimensional, constant-density stagnation point flow with imbedded freestream turbulence, an analytical solution has been found to link the outer flow with an inner viscous layer flow. The physical demand as the stagnation point is approached is that production is in balance with dissipation. However, it is found that, to avoid a singularity, one of the conventional constants in the dissipation rate equation must be mildly changed. The final demonstration is that an analytical relation exists between the stagnation point and freestream "point" turbulence quantities and that the stagnation point values are not simply proportional to the freestream values of k and ϵ.

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The Decay of the Shock Wave from a Supersonic Projectile

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Introduction

It is well known that the sound wave from a supersonic projectile of any shape develops into an N wave after having travelled some distance from the source. This means that the wave profile in time consists of a front shock and a linear time dependence ending with a tail shock. The calculation of the asymptotic decay of these two shocks of equal magnitude appears in the literature, for example, in works by Lighthill,[1] Pierce,[2] and Whitham.[3] It is found that the shock wave amplitude near the Mach cone decreases as $r^{-3/4}$, where r is the radial distance between the flight path and the point of observation. The textbook derivation of this interesting inverse three-quarters power law of asymptotoic decay of cylindrical shock waves is based on nonlinear geometrical acoustics theory with a boundary condition at a distance from the source where linear acoustic theory is still valid.

However, the decay of the shock does not only mean that the shock amplitude decreases, but also that the shock width grows so that the shock finally vanishes. This vanishing of the shock cannot be accounted for by the inviscid wave theory used in Refs. 1-3. In order to obtain the correct wave profile at distances from the source where the shock width has grown so that the shock vanishes, it is necessary to consider dissipative effects, i.e., viscosity and heat conduction. In this Note, instead of using geometrical acoustics, a generalized Burgers equation for cylindrical waves is used. The derivation is similar to that made by the present author in a recent study of nonlinear sound waves from a uniformly moving sinusoidal source.[4] The Mach number of the source is a parameter in the generalized Burgers equation, and it is shown that this equation gives the usual $r^{-3/4}$ decay of the shock wave if the dissipative term is neglected. The development of a cylindrical N wave into a smooth wave profile according to a generalized Burgers equation has been investigated by the present author.[5] The result of this investigation is used herein to give the amplitude and the smooth profile of the sound pulse from a supersonic projectile at distances so far from the source that the shock has broadened and disappeared.

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Derivation of a Generalized Burgers Equation

A useful starting point for the derivation of wave equations in nonlinear acoustics is the equation given by Kuznetsov,[6] correct to the second order in the small quantities ϕ and b:

$$\frac{\partial^2\Phi}{\partial t^2}-c^2\Delta\Phi=\frac{\partial}{\partial t}\left\{(\nabla\Phi)^2+\frac{b}{\rho_0}\Delta\Phi+\frac{\gamma-1}{2c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2\right\} \tag{1}$$

where Φ is the velocity potential, i.e., $v=-\text{grad}\,\Phi$,

$$b=\frac{4}{3}\eta+\zeta+\kappa\left(\frac{1}{c_v}-\frac{1}{c_p}\right) \tag{2}$$

where ζ and η are the shear and bulk viscositites, κ the coefficient of thermal conductivity, and c_v and c_p the heat capacities of the fluid, ρ_0 the density, and c the wave velocity of the undisturbed fluid.

It is assumed that the projectile travels along the x axis with the velocity $V>c$. We introduce the following new coordinates, assuming cylindrical symmetry:

$$X=c[t-(V/c^2)x] \tag{3}$$

$$r=\sqrt{y^2+z^2} \tag{4}$$

$$T=(x-Vt)/c \tag{5}$$

The linear acoustic theory of sound from a supersonic projectile in Ref. 2 is based upon the solution of the equation

$$\frac{\partial^2\Phi}{\partial t^2}-c^2\Delta\Phi=c^2\dot{A}(Vt-x)\delta(y)\delta(z) \tag{6}$$

where $A(Vt-x)$ is a function depending on the geometrical form of the projectile and

$$A(\xi)\neq 0 \text{ for } 0<\xi<L \tag{7}$$

As in the linear case, we want to find a solution to Eq. (1) that depends only on r and T. Insertion of Eqs. (3-5) into Eq. (1) and use of the condition

$$\frac{\partial\Phi(X,r,T)}{\partial X}=0 \tag{8}$$

give

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}-\frac{M^2-1}{c^2}\frac{\partial^2\Phi}{\partial T^2}$$
$$=\frac{2V}{c^5}\left(1+\frac{\gamma-1}{2}M^2\right)\frac{\partial\Phi}{\partial T}\frac{\partial^2\Phi}{\partial T^2}+\frac{V}{c^5}\frac{b}{\rho_0}\frac{\partial^3\Phi}{\partial T^3}$$
$$+\frac{V}{c^3}\frac{\partial}{\partial T}\left(\frac{\partial\Phi}{\partial r}\right)^2+\frac{b}{\rho_0}\frac{V}{c^3}\frac{\partial}{\partial T}\left\{\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right\} \tag{9}$$

where the Mach number $M=V/c$.

Now new variables are introduced in Eq. (9):

$$\tau=-T-\frac{r-r_0}{c}(M^2-1)^{1/2} \tag{10}$$

$$\xi=\nu r \tag{11}$$

where r_0 is the radial distance from the flight path where linear acoustic theory is still valid and ν is a small dimensionless parameter. Inserting Eqs. (10) and (11) into Eq. (9) and expanding to second order in the small quantities ν, Φ, and b, we obtain a generalized Burgers equation:

$$\frac{\partial v}{\partial r}+\frac{1}{2}\frac{v}{r}-\frac{M^3}{c^2(M^2-1)}\frac{\gamma+1}{2}v\frac{\partial v}{\partial\tau}$$
$$-\frac{M^3 b}{2c^3\rho_0\sqrt{M^2-1}}\frac{\partial^2 v}{\partial\tau^2}=0 \tag{12}$$

where

$$v=v_r=-\frac{\partial\Phi}{\partial r}\approx\frac{\sqrt{M^2-1}}{c}\frac{\partial\Phi}{\partial\tau} \tag{13}$$

The Decay of the Shock Wave

We will first verify that the textbook expression for the decay of the shock wave is obtained from Eq. (12). The change of variables

$$U=v(r/r_0)^{1/2} \tag{14}$$

$$\zeta=2(rr_0)^{1/2} \tag{15}$$

transforms Eq. (12) into

$$\frac{\partial U}{\partial\zeta}-\frac{M^3}{M^2-1}\frac{\gamma+1}{2c^2}U\frac{\partial U}{\partial\tau}=\frac{M^3}{\sqrt{M^2-1}}\frac{b\zeta}{4c^3\rho_0 r_0}\frac{\partial^2 U}{\partial\tau^2} \tag{16}$$

If the right-hand side of Eq. (16) is neglected and the remaining equation is solved with the boundary condition

$$U(\zeta=2r_0,\tau)=G(\tau) \tag{17}$$

where $G(\tau)$ is obtained from Eq. (6) as in Ref. 2, then the standard technique for obtaining the front shock amplitude gives the same result as in Ref. 2, Eq. (11-10.18). We must only remember that the relation between v and the acoustic pressure p is

$$v=\frac{p}{\rho_0 c}\frac{\sqrt{M^2-1}}{M} \tag{18}$$

In order to study Eq. (12) for that stage of the evolution of the wave where the N wave has become fully developed, we introduce the dimensionless variables according to Crighton and Scott[7]

$$W=\frac{v}{v_0}\left(\frac{r}{R_0}\right)^{1/2} \tag{19}$$

$$s=-\frac{1}{\ell_0}\{cT+(r-R_0)(M^2-1)^{1/2}\} \tag{20}$$

$$\eta=1+\frac{M^3}{M^2-1}(\gamma+1)v_0\frac{R_0^{1/2}r^{1/2}-R_0}{c_0\ell_0} \tag{21}$$

and the dimensionless constant

$$\eta_0=\frac{M^3}{M^2-1}(\gamma+1)v_0\frac{R_0}{c_0\ell_0} \tag{22}$$

Here R_0 is the radial distance from the flight path at which the N wave is fully developed and $2\ell_0/c$ is the duration of this N wave pulse. The generalized Burgers equation for W

then is

$$\frac{\partial W}{\partial \eta} - W\frac{\partial W}{\partial s} = \epsilon\frac{\eta - 1 + \eta_0}{2}\frac{\partial^2 W}{\partial s^2} \tag{23}$$

$$\epsilon = \frac{2b\sqrt{M^2-1}}{\rho_0 v_0 (\gamma+1)\ell_0 \eta_0} \tag{24}$$

The N wave solution of Eq. (23) is obtained by setting $\epsilon = 0$:

$$\begin{aligned} &W(s,\eta) = -(s/\eta), \ s<\eta^{1/2}; \qquad W(s,1) = -s, \ s<1 \\ &W(s,\eta) = 0, \qquad s>\eta^{1/2}; \qquad W(s,1) = 0, \quad s>1 \end{aligned} \tag{25}$$

In Eqs. (25) we have assumed $s>0$ because of the antisymmetry of $W(s,\eta)$ about $s=0$. The detailed behavior of $W(s,\eta)$ in the neighborhood of $s=\eta^{1/2}$ is from Ref. 7,

$$W(s,\eta) = -\frac{1}{2}\eta^{-1/2} \times \left\{1 - \tanh\frac{[(s-\eta^{1/2})/\epsilon] - \frac{1}{2}\eta^{1/2}(\eta - 1 + (\eta_0 - 1)\ell n\eta)}{4\eta^{1/2}[(\eta+\eta_0-1)/2]}\right\} \tag{26}$$

The problem of finding a solution of Eq. (23) for large η values that matches solutions (25) and (26) is solved in Ref. 5, i.e.,

$$W = -C\frac{s}{\epsilon\eta^2}\Phi\left(1,\frac{3}{2};-\frac{s^2}{\epsilon\eta^2}\right) \tag{27}$$

where

$$C = 1 - \tanh\frac{1}{4} \tag{28}$$

and Φ is the confluent hypergeometric function. With the physical variables we obtain the following expression for v, which also gives us the pressure p through Eq. (18):

$$v = -Cr^{-3/2}s\frac{\ell_0}{c}c^2R_0^{1/2}\frac{\rho_0\ell_0 v_0}{2b}\frac{(M^2-1)^{1/2}}{M^3} \times \Phi\left(1,\frac{3}{2};-\left(\frac{s\ell_0}{c}\right)^2\frac{c^3\rho_0}{2br}\frac{(M^2-1)^{1/2}}{M^3}\right) \tag{29}$$

(In Ref. 5, formula (50), ϵ is missing in the denominator of the argument of Φ.) According to Eq. (29) the s maximum of v decays as r^{-1}. The order of magnitude of r in Eq. (29) is

$$\frac{c_0^2\ell_0^2}{v_0^2 R_0 r} = o(\epsilon^2) \tag{30}$$

The quantities v_0 and R_0 in Eq. (29) need some comment. As was mentioned before, for $r=R_0$ the N wave is fully developed. Since for a fully developed N wave the shock amplitude v_0 decreases with $R_0^{-1/4}$, the old-age value of v in Eq. (29), the right-hand side of which contains the product $v_0R_0^{1/2}$, thus decreases with $R_0^{-1/4}$. Thus it is desirable to determine the value of R_0 as exactly as possible. This can be done by the method in Ref. 2 in the following way. The value of v_0, according to Ref. 2, formula (11-10.18) and Eq. (18) herein, is

$$v_0 = \left\{\left[2^{1/4}\left(\mathrm{Max}\int_{-\infty}^{\xi}F_W(x)\,\mathrm{d}x\right)^{1/2}c(M^2-1)^{-1/8}\right] \div [R_0^{1/2}(R_0-r_0)^{1/4}(\gamma+1)^{1/2}]\right\}\frac{\sqrt{M^2-1}}{M} \tag{31}$$

where $F_W(x)$ is the Whitham F function, which is known if the form of the projectile is known. The limit ξ in this integral depends on R_0 and the N wave can be considered as developed when R_0 has reached a value for which the integral over F_W in Eq. (31) does not grow essentially greater. An uncertainty of 50% in the determination of R_0 gives an uncertainty of about 10% in v in the result [Eq. (29)].

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A Noniterative Finite Difference Method for the Compressible Unsteady Laminar Boundary Layer

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Introduction

In order to determine the friction drag and the rate of heat transfer at the surface of a body in dynamic motion, the unsteady viscous flow must be investigated in detail. Although major progress has been made over the decades in the study of unsteady viscous flows, our information is still not sufficient, due to the many flow-complicating factors. Even for the simpler attached boundary layers, the problem can still be complicated through interaction among the increased dimensionality, nonlinearity, and compressibility.

A variety of successful methods have appeared in the literature for general, unsteady boundary-layer research. A comprehensive list is found in Tellionis.[1] In numerical simulation of the unsteady boundary layers, iterative methods have been widely used due to the nonlinear convection terms in the governing equations.[2,6]

In this Note, the nonlinearity is eliminated by using a known technique of linearizing the general nonlinear implicit finite difference equations without sacrificing accuracy.[3,4] Transient as well as oscillating two-dimensional compressible boundary layers are solved numerically by using a non-

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